

Nested strange attractors in spatiotemporal chaotic systems

L. S. Tsimring

Institute for Nonlinear Science, University of California, San Diego, California 92093-0402

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A self-similar fractal structure for phase-space attractors is observed for time series produced by spatiotemporal chaotic systems. Two data sets, produced by (1) coupled logistic maps and (2) the complex Ginzburg-Landau equation, are studied numerically. The attractor reconstructed in a time-delay embedding space has a coarse-grained dimension growing exponentially with increasing resolution. A coarse-grained K_2 entropy in the region of scaling grows linearly with the embedding dimension. This type of scaling behavior is expected for developed spatiotemporal chaos in spatially homogeneous extended systems when the correlation length is much smaller than the system size. The growth rate of the dimension (differential dimension) is proportional to a density of dimensions and a correlation length of the system. The growth rate of K_2 entropy is proportional to the entropy density and the correlation length.

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I. INTRODUCTION

Spatially extended chaotic systems are an important type of deterministic systems with many degrees of freedom. There is little hope to develop a general quantitative description of multidimensional systems with the same details as for low-dimensional chaotic systems. However, the spatial homogeneity of extended systems provides some additional symmetry and thus some universal properties of its high-dimensional attractor. There is numerical and analytical evidence that, as the system length L tends to infinity, the spectrum of Lyapunov exponents acquires a universal form, although the number of positive Lyapunov exponents goes to infinity [1-3]. It follows from this that at least the Lyapunov dimension D_L of spatially homogeneous systems is an extensive quantity and hence the density of dimension $\rho_L = D_L/L$ is independent of L . The same property can be expected for other fractal dimensions. The density of Lyapunov dimension has been calculated for several simple model systems, such as short coupled logistic map lattices [1-3], in a straightforward manner by evaluating the overall Lyapunov dimension and dividing it by the system length. Unfortunately, for a long system the straightforward approach seems unfeasible and the question of estimating the density of dimension remains. In situations when the equations of motion are unknown and one deals with the data only, the problem of computing Lyapunov exponents gets even more difficult [4]. Some authors [2,6] argue that the density of dimension can be found by computing a dimension in an embedding space of vectors \mathbf{y}_s constructed using a space-delay embedding from the spatiotemporal data $y(x, t)$,

$$\mathbf{y}_s = \{y(x, t), y(x + \Delta, t), \\ y(x + 2\Delta, t), \dots, y(x + (d_E - 1)\Delta, t)\}. \quad (1)$$

The idea was that for large d_E , the dimension grows asymptotically as $\rho_D d_E$. Torcini *et al.* [3] pointed out

that this is not strictly correct, since at finite d_E one deals with the projection of a high-dimensional attractor onto a low-dimensional subspace and generically the dimension of the projection has to be equal to the embedding dimension. However, a nontrivial scaling relation between a projection dimension and the embedding dimension has been recovered [3,5] for the *coarse-grained* information dimension

$$D_{CG}^{inf}(\delta) \equiv -\frac{d \ln n}{d \ln \delta} \simeq \rho_D d_E, \quad \rho_D < 1 \quad (2)$$

within some finite range of δ . Here n is the number of data vectors and δ is the distance between a reference point and its nearest neighbor in the embedding space [7]. A reasonable explanation of this fact (see [5]) is that for d_E large enough a piece of length $(d_E - 1)\Delta$ evolves almost as an isolated system with a weak $O(d_E^{-1/2})$ contact through a "thermal bath" with the rest of the system. Then, for a range of scales much less than the size of attractor S_A and larger than a scale of "external noise" $S_A d_E^{-1/2}$ one can expect the value of a coarse-grained dimension to be a constant proportional to d_E , as it happens for isolated map chains, when the dimension is proportional to the length of the chain (see [1,3]). The coefficient of proportionality then is presumably a density of dimension. However, numerically it turns out that such a scaling appears for a range of embedding dimensions starting at about 6 or higher. To estimate ρ_D Politi and Puccioni [5] employed embedding dimension as large as 14, which requires an enormous amount of data and supercomputer power. It shows that for practical estimations this scaling relation is of little use.

In the present paper we demonstrate that another important scaling relation exists for spatially homogeneous systems revealing spatiotemporal chaos. This scaling is observed in the structure of an attractor in a high-dimensional embedding space of vectors constructed in the usual manner from time-delayed values of the field

variable at one fixed space location

$$\mathbf{y}_t = \{y(t, x), y(t + T, x), \\ y(t + 2T, x), \dots, y(t + (d_E - 1)T, x)\}. \quad (3)$$

It turns out that the coarse-grained correlation dimension of the attractor in this space is a linear function of the logarithm of scale, so the smaller the scale, the greater the coarse-grained dimension (this growth is saturated, of course, when D_{CG} approaches d_E). We propose a possible explanation of this phenomenon of scaling in which the coefficient of proportionality is related to the density of dimension and the correlation length of the system.

II. NUMERICAL COMPUTATIONS

In our numerical computations we used two spatiotemporal dynamical systems as generators of data: (a) the coupled-map lattice (CML)

$$x(n + 1, j) = f(x(n, j)) \\ + D[f(x(n, j - 1)) - 2f(x(n, j)) \\ + f(x(n, j + 1))] \quad (4)$$

(b) and the one-dimensional complex Ginzburg-Landau equation (CGLE)

$$\partial_t A = \varepsilon A + (1 + ic_1)\partial_x^2 A - (1 - ic_3)|A|^2 A. \quad (5)$$

For the first model we used a chain of 1000 maps with periodic boundary and random initial conditions, the logistic function $f(y) = 4y(1 - y)$ for local dynamics, and several values of coupling $D = 0.05, 0.2, 0.5, \frac{2}{3}$. For the second model (CGLE) we employed a pseudospectral code [9] with 1024 spectral harmonics and periodic boundary conditions. The coefficients c_1, c_3 , and ε of CGLE (5) were chosen in order to satisfy the conditions of the defect turbulence (see [10,11]): $c_1 = 3.5, c_3 = 0.9$, and $\varepsilon = 1.0$. The physical length of the system was taken to be $L = 500$, and the time step both for integration and the output was 0.1. We generated long streams of accurate data (200 000 points of double-precision numbers in each) for the values

$$y(n) = x(n, 500) \quad (6)$$

for CML and

$$y(n) = \text{Re}A(0.1n, 250.0) \quad (7)$$

for CGLE. Then we reconstructed the phase space via usual time-delay embedding (3) and computed the correlation integrals of the attractor. We found the value of time delay T using the mutual information criteria [12], namely the value corresponding to the first minimum of the average mutual information

$$I(T) = \int \int dy_1 dy_2 p_{t,t+T}(y_1, y_2) \ln \frac{p_{t,t+T}(y_1, y_2)}{p_t(y_1)p_{t+T}(y_2)}, \quad (8)$$

where $p_t(y)$ is the probability of having a value y at the time t and $p_{t,t+T}(y_1, y_2)$ is the joint probability of finding y_1 at t and y_2 at $t + T$. For CML, the time delay

was always 1, and for CGLE, $T = 2.5$. For the computation of the correlation integral we used the Grassberger-Procaccia algorithm [13]:

$$C(r; d_E) = \frac{1}{(N - 2W - 1)M} \\ \times \sum_{i=1}^M \sum_{j=1, |i-j|>W}^N \Theta(r - \|\mathbf{y}(i) - \mathbf{y}(j)\|_\infty), \quad (9)$$

where the L_∞ norm $\|\mathbf{x}\|_\infty$ means the maximal scalar component of the vector \mathbf{x} and Θ is the Heaviside function. N is the total number of data points and M is the number of reference points, which was usually taken to be $0.1N$. An appropriate choice of W reduces the spoiling effect of autocorrelations from the correlation sums as suggested by Theiler [8]. Using the L_∞ norm allowed us to construct a quite efficient numerical code. Still, to keep a reasonable balance between good statis-

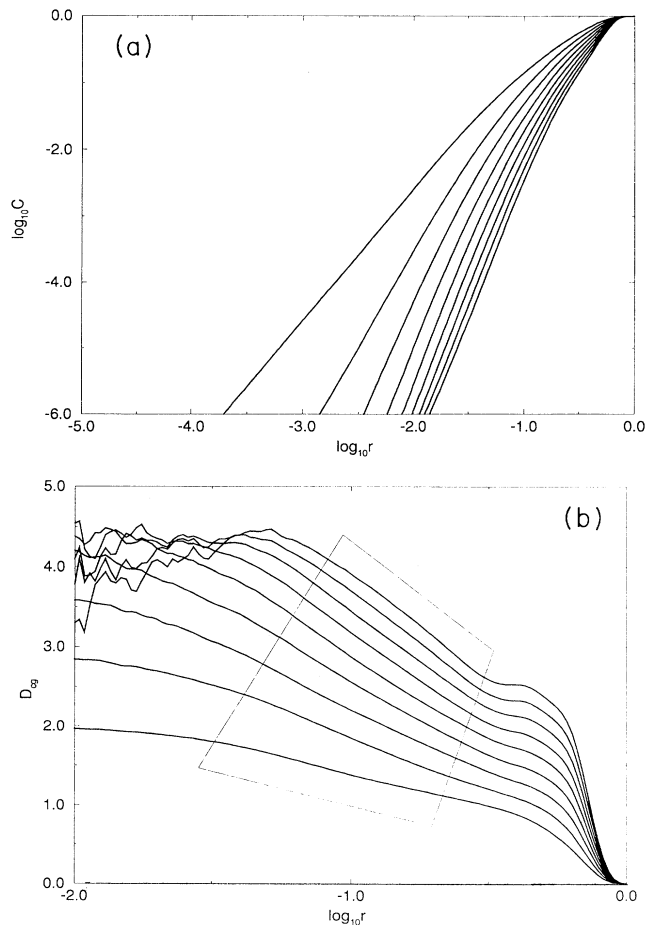


FIG. 1. (a) Correlation integrals and (b) coarse-grained correlation dimensions for the coupled-map lattice (4) for $a = 4.0$ and $D = \frac{2}{3}$ at different embedding dimensions $d_E = 2, \dots, 10$ for 200 000 data points. Solid polygon shows the range of points taken for the calculation of differential dimension S and parameter P .

tics and the computational time, we used intermediate values of embedding dimension $d_E = 2, \dots, 10$ in all computations (with 200 000 data points and 20 000 reference points each calculation of a set of correlation integrals took about 20 h of Sparcstation 2 computer CPU time).

The results of computations are presented in Figs. 1 and 2. Figure 1(a) shows the decimal logarithms of the correlation integrals $\log_{10} C(r; d_E)$ as a function of $\log_{10} r$ for the data stream generated by CML with $D = \frac{2}{3}$. For high enough embedding dimensions $d_E \geq 5$ they have a form of convex lines which turn out to be parabolas over a wide range of scales. That is evident from Fig. 1(b), where coarse-grained correlation dimensions $D_{CG} = d \log_{10} C / d \log_{10} r$ are shown as functions of $\log_{10} r$ for different d_E . As one can see, D_{CG} grows linearly with $\log_{10} r$,

$$D_{CG} \simeq D_0 - S \log_{10} r, \quad (10)$$

with approximately the same slope $S \simeq 2.7$, independently of d_E . Remarkably for large enough d_E the value of D_0 is itself asymptotically a linear function of embed-

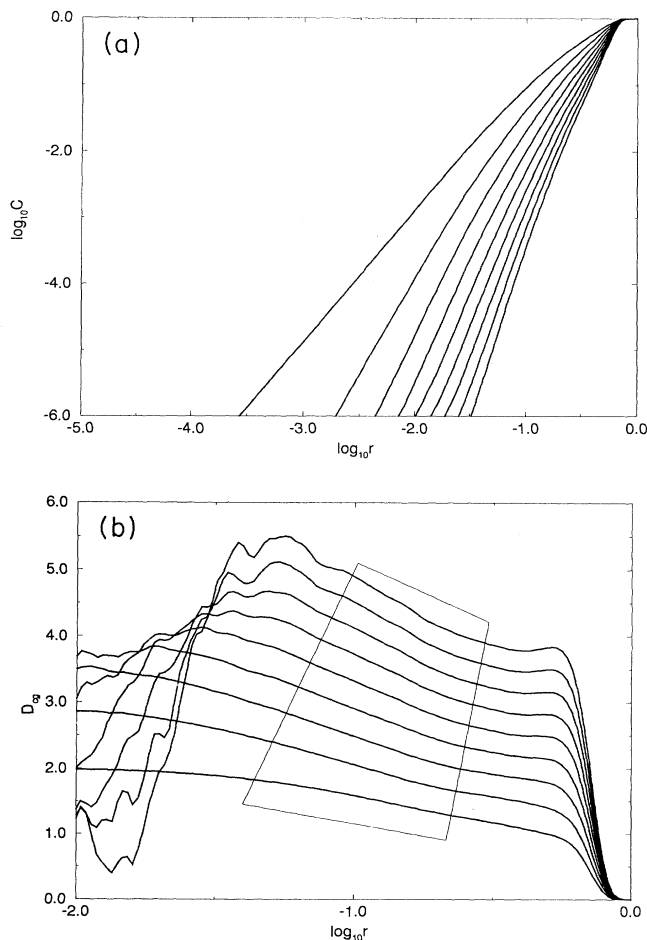


FIG. 2. The same as Fig. 1 for CGLE (5) with $c_1 = 3.5, c_3 = 0.9, \epsilon = 1.0, L = 500, N = 2 \times 10^5$, and $T = 2.0$.

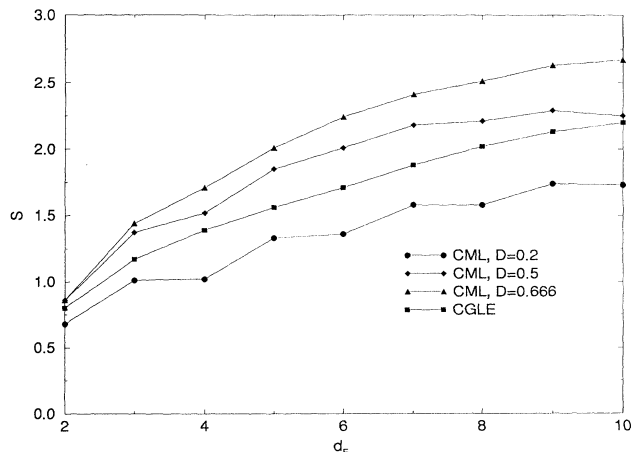


FIG. 3. Differential dimension S as a function of embedding for dimension d_E for the coupled-map lattice with different diffusion constants and complex Ginzburg-Landau equation.

ding dimension

$$D_0(d_E) - D_0(d_E - 1) \simeq P \quad (11)$$

with $P \simeq 0.22$. We repeated the computations for other values of the diffusion coefficient $D = 0.5, 0.2$ with essentially the same result, although the numerical values of S and P were different. At small D correlation integrals lose smoothness and acquire a piecewise linear structure (see Sec. III). Moreover, the same kind of universal behavior is clearly seen in Fig. 2 for CGLE. In Figs. 3 and 4 the dependences of S and P on d_E are shown for our numerical models.

To check the sensitivity of our numerical results to the parameters of computations, we calculated correlation integrals for CGLE with different lengths of the system ($L = 500, 1000$), different numbers of points in the time series ($N = 2 \times 10^5, 10^5, 5 \times 10^4, 2 \times 10^4$), and different

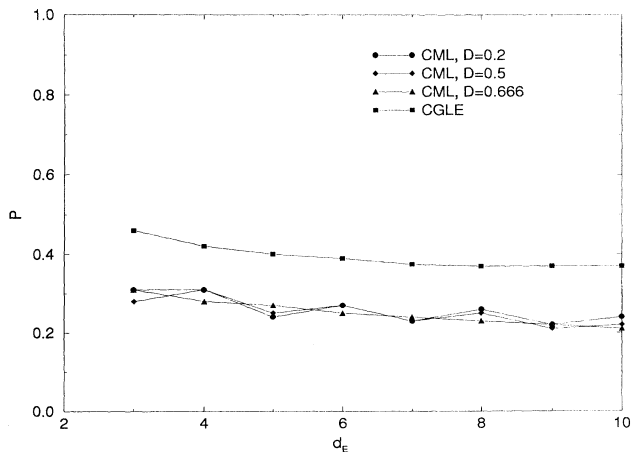


FIG. 4. Parameter P (differential K_2 entropy) as a function of embedding dimension d_E .

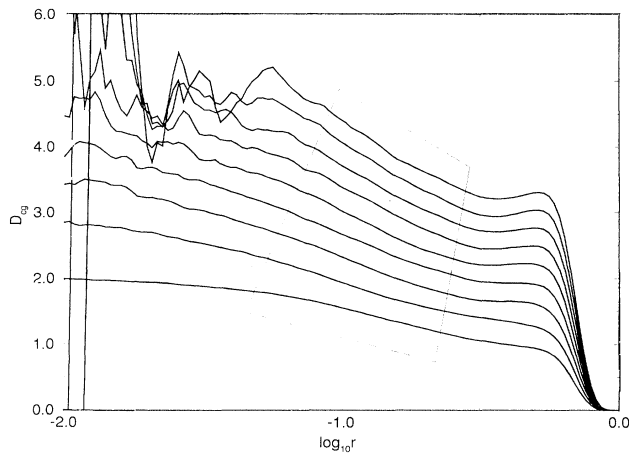


FIG. 5. The same as Fig. 2(b) for CGLE with $L = 1000$ and with different number of points $N = 2 \times 10^4$ (all other parameters are the same).

time delays ($T = 2.0, 2.5$). A plot of the coarse-grained dimensions for $L = 1000, T = 2.0$, and $N = 2 \times 10^4$ is shown in the Fig. 5. The scaling behavior clearly persists, however, due to fewer data points as compared to the case of Fig. 2, and the band of scaling gets somewhat smaller. Plots of differential dimension S versus d_E remain close to each other within the 10–15% range (see Fig. 6). However, one still can notice somewhat lower values of S for $L = 500$, which can be cause by finite-size effects.

For another test, we made analogous computations of correlation integrals and coarse-grained dimensions for white noise, uniformly distributed between 0 and 1. The results of computations are shown in the Fig. 7. As one can see, there is no scaling region with constant slope of coarse-grained dimension at all in this case.

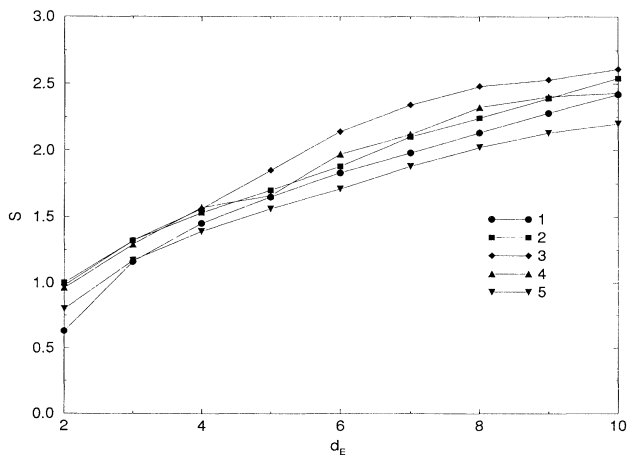


FIG. 6. Differential dimension S as a function of embedding dimension for CGLE (5) 1- $N = 2 \times 10^5$, $T = 2.0$, $L = 1000$, 2- $N = 10^5$, $T = 2.0$, $L = 1000$, 3- $N = 5 \times 10^4$, $T = 2.5$, $L = 1000$, 4- $N = 2 \times 10^4$, $T = 2.0$, $L = 1000$, 5- $N = 2 \times 10^5$, $T = 2.0$, and $L = 500$ (the same as in Fig. 3).

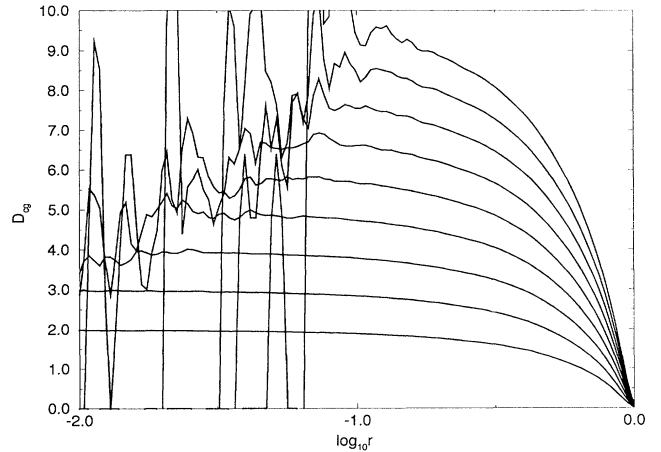


FIG. 7. The same as Fig. 1(b) for the white noise uniformly distributed between 0 and 1 for 200 000 data points.

These computations demonstrate that the attractor of our spatiotemporal systems is organized in a universal manner. As the length of a fractal line grows with increasing resolution, in the present case the dimension itself grows as we increase the resolution in the phase space. We wish to call such objects nested strange attractors to highlight their striking difference from a “normal” strange attractor with a fixed value of fractal dimension independent of resolution, on one side, and from purely random sequences, on the other.

III. DISCUSSION

In this section we present a plausible explanation of the scaling observed in our numerical experiments and show how it can be applied for the estimation of the density of dimension. Actually, the relevant arguments have already been mentioned in the paper by Torcini *et al.* [3] while discussing the paper by Mayer-Kress and Kurz [14]. Indeed, one may assume that for developed spatiotemporal chaos a finite correlation length exists which in the simplest case determines a rate of exponential decay of the space correlation function:

$$\langle (y(x, t) - \langle y \rangle)[y(x + X, t) - \langle y \rangle] \rangle \propto 10^{-X/l_c}. \quad (12)$$

Here l_c is the correlation length and $\langle \rangle$ means ensemble averaging. A number of other methods to find correlation lengths in spatiotemporal chaotic systems have also been devised (see the review [15]). Since we deal with nonlinear systems, probably a better characterization of the space correlations is presented by the average mutual information $I(X)$, which is defined as follows [12]:

$$\begin{aligned} I(X) &\equiv I(y(x, t), y(x + X, t)) \\ &= \int \int dy_1 dy_2 p_{x, x+X}(y_1, y_2) \ln \frac{p_{x, x+X}(y_1, y_2)}{p_{x+X}(y_1) p_x(y_2)}, \end{aligned} \quad (13)$$

where $p_x(y_1)$ is the probability of having a value y_1 at

location x [similarly for $p_{x+X}(y_2)$] and $p_{x,x+X}(y_1, y_2)$ is the joint probability of finding y_1 at x and y_2 at $x+X$. In spatiotemporal chaotic systems the mutual information also decays exponentially in space: $I(X) \simeq 10^{-X/l_c}$. The correlation length can be estimated as v/λ_{\max} , where v is the velocity of the information transport [2,16] and λ_{\max} is the maximal Lyapunov exponent of the system. It means that the attractor reconstructed out of the time series taken at space location x_0 will have no information about the dynamics outside the interval $[x_0 - X, x_0 + X]$ when it is coarse grained with the scale $r_X = 10^{-X/l_c}$ or larger. More precisely, there should exist a time series $\bar{y}(x_0, t)$ corresponding to a finite-dimensional attractor of the subsystem of the length X such that $|\bar{y}(x_0, t) - y(x_0, t)| \leq Cr_X$ (C is constant). This is a rather mathematical statement, which we are unable to prove rigorously, but it seems feasible from the physical point of view. The finer the resolution, the larger the interval responsible for the attractor structure and therefore the larger the dimension one should measure. Increasing the phase-space resolution by one order of magnitude effectively lengthens the part of the system contributing to the correlation integral by $2l_c$. Accordingly, the coarse-grained correlation dimension would grow by $2\rho_D l_c$. In general, one can expect the following simple relation between the coarse-grained correlation dimension and the resolution scale (hypercube size) r :

$$D_{CG} \simeq D_0 - 2\rho_D l_c \log_{10} r. \quad (14)$$

Note that (14) has the same functional form as (10) and thus $S = 2\rho_D l_c$. Therefore, independent numerical computations of the constant S and the correlation length l_c allow us to estimate the density of dimension $\rho_D \simeq S/2l_c$. For a coupled-map lattice with $D = \frac{2}{3}$ we found $l_c \simeq 4.0$ and $S = 2.7$ (see Fig. 3), therefore $\rho_D \simeq 0.34$. For CGLE $l_c = 25.0$ and $S \geq 2.25$, so $\rho_D \geq 0.05$.

It is well known for low-dimensional strange attractors [13] that in the limit $d \rightarrow \infty$ and $r \rightarrow 0$

$$\log_{10} C(r; d_E) = -D_2 \log_{10} r + K_2 T d_E, \quad (15)$$

where D_2 is the correlation dimension of the attractor and K_2 is an entropylike quantity which is a lower bound for the Kolmogorov-Sinai entropy. In practice we have only a finite amount of data and only an approximate relation for intermediate scales and embedding dimensions holds

$$\log_{10} C(r; d_E) = - \int^{\log_{10} r} D_{CG} d(\log_{10} r') + K_{CG} T d_E, \quad (16)$$

where we introduced a *coarse-grained* K_2 entropy K_{CG} . In spatiotemporal chaos not only D_2 is expected to be an extensive quantity, but also K_2 [2]. Then the same arguments lead us to the conjecture that the coarse-grained K_2 entropy has to be proportional to $\log_{10} r$ and l_c , i.e.,

$$\log_{10} C(r; d_E) = \rho_D l_c (\log_{10} r)^2 - 2\rho_K l_c T d_E \log_{10} r - A \log_{10} r + B \quad (17)$$

or

$$D_{CG} = -2\rho_D l_c \log_{10} r + 2\rho_K l_c T d_E + A, \quad (18)$$

where A and B are some constants. The last formula explains the linear dependence of D_0 on d_E and gives us the opportunity to compute the density of K_2 entropy for spatiotemporal systems. For our examples (see Fig. 4) (a) CML $\rho_K \simeq 0.85$ and (b) CGLE $\rho_K \simeq 0.003$.

The scaling of the correlation integral does not necessarily have such a clean form as in these examples. In general, it can be masked by internal structures due to inhomogeneity (“lacunarity” [8]) of an attractor and can appear only after averaging them out. An example of this type of behavior is presented in Fig. 8 for coupled-map lattice (5) at small diffusion $D = 0.01$. In this case instead of a smooth line we have a piecewise linear correlation integral. The first piece with a slope close to 1.0 corresponds to the one-map dynamics (the impact of all other maps is not resolved at those scales), the next piece has a slope close to 3.0, which corresponds to a chain of three maps, etc. The scaling (14) would be seen only in an averaged form for much larger amounts of data.

In this paper we considered one-dimensional spatially extended systems only. If our explanation of the scaling is correct, one can determine the space dimension observing just the time series in one space point. Indeed, for two-dimensional space one can expect a cubic dependence of the correlation integral on the logarithm of scale, a quartic scaling for three-dimensional systems, and so on. However, this conjecture has not been tested numerically yet.

In conclusion, let us note that the mechanism described above forming the self-similar structure of the nested strange attractor is only a qualitative interpretation of our numerical findings. In the present work we have tested only two data sets generated by a coupled-map lattice and the Gizburg-Landau equation; however, we believe that the observed scaling is a generic property of homogeneous spatiotemporal chaotic dynamics. In any case, further numerical and theoretical work are needed to determine the extent of universality of this phenomenon in various extended chaotic systems.

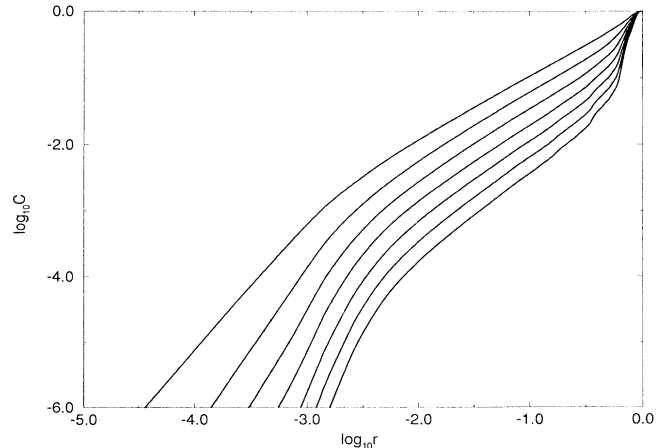


FIG. 8. Correlation integrals for CML with $D = 0.01$ at embedding dimensions $d_E = 2, \dots, 8$ for 200 000 data points.

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